AIAA 82-4093

Unsteady Response of Rectangular Wings in Spanwise Uniform Shear Flow

Makoto Kobayakawa*
Kyoto University, Kyoto, Japan

In engineering applications, it is important to estimate aerodynamic forces on wings in nonuniform flow. Wakes produced by objects ahead of wings frequently affect the performance of airplanes, turbomachines, etc. In this paper, a theory of wings oscillating in shear flow is presented. The whole flowfield is assumed to be inviscid and incompressible, and, furthermore, the shear flow velocity is assumed to vary linearly along the wing span. For this problem, lifting surface theory is applied. An integral equation, which is similar to that for potential flow, is derived under the assumption of small perturbations, and then solved numerically by the mode function method. Generalized forces, Q_{ij}, which can be easily related to unsteady lift forces and moments, are obtained; moreover, response functions of wings to sinusoidal heaving oscillations and a sinusoidal gust are calculated for three different amounts of shear flow. The results are compared for several frequencies and show that the effect of shear is not large except in the limiting case. However, the generalized forces and gust response functions are affected by the shear to some extent, even when the degree of shear is moderate. The effect increases as the frequency becomes larger.

Nomenclature

$c(y)$	= chord length at y
$\bar{c}$	= mean chord length
D,d	= reference area and reference length
i,j	= force mode and downwash mode
$K(x,y;x',y')$	= kernel function
l	= lift distribution
m	= number of spanwise collocations
N	= number of chordwise collocations
p	= pressure
Q_{ij}	= generalized force, $Q_{ij}=Q'_{ij}+i\bar{\nu}Q''_{ij}$
Q'_{ij},Q''_{ij}	= stiffness derivative and damping derivative
R	$=\sqrt{(x-x')^2+(y-y')^2+z^2}$
R_1	$=\sqrt{(\lambda-x')^2+(y''-y')^2+z^2}$
R_2	$=\sqrt{\lambda^2+(y''-y')^2+z^2}$
S	= wing surface
s	= half-span width
u,v,w	= perturbation velocities
$U(y)$	= shear flow velocity distribution, $U(y)=U_0(1+y/e)$
V	= perturbation acceleration
W	= wake
$x_l(y)$	= position of leading edge
x,y,z	= Cartesian coordinates
Γ_q	= mode function along spanwise direction, Eq. (20)
Δ	= difference between upper and lower surfaces
η	$=y/s$
$\bar{\nu}(y)$	$=\omega\bar{c}/U(y)$
ρ	= flow density
Ψ_q	= mode function along chordwise direction
ω	= circular frequency of oscillation

Introduction

NEEDLESS to say, one of the most important problems in aeronautical engineering and its applications is the calculation of aerodynamic forces, such as lift force or the pitching moment of finite wings, especially in unsteady motion and in gusty winds. A number of papers have been published on this problem in unsteady lifting surface theory (e.g., Refs. 1 and 2). In almost all of these investigations, uniform flow (potential flow) is assumed. In engineering applications, however, the upstream conditions of wings are not always uniform. Airplanes encounter "wind shear" at takeoff and landing, while helicopter rotor blades, turbine blades, etc., are frequently submerged in the wake of the preceding blade. Moreover, flow against helicopter rotor blades and turbine blades always behaves in the same way as shear flow along the spanwise direction, and the wings of an airplane engaged in a steep turn have the same effect as rotating blades. Accordingly, it is important to estimate the aerodynamic forces on wings in nonuniform flow.

Some theoretical studies of the problem of wings in shear flow along the spanwise direction have been published. In Ref. 3, von Kármán and Tsien used the lifting line theory when investigating a finite wing that encounters three-dimensional shear flow. Honda[4] analyzed the characteristics of a wing between parallel walls facing the flow with spanwise nonuniform velocity distribution. Lighthill[5] used the Fourier transform to obtain the general solutions for small disturbances to two-dimensional parallel shear flow. Morita succeeded in calculating the lift distribution of small aspect ratio wings in shear flow by applying the Jones theory,[6] and also obtained the lift distribution of large aspect ratio wings by using the lifting line theory.[7] However, most of these in-

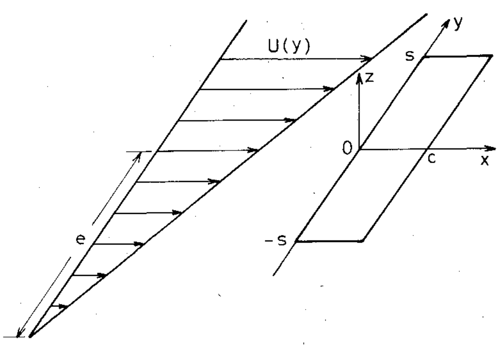

Fig. 1 Wing in uniform shear flow and system of coordinates.

Received March 2, 1981; revision received Aug. 4, 1981.

*Associate Professor, Dept. of Aeronautical Engineering. Member AIAA.

vestigations treated steady cases. The author has presented the results of calculations of unsteady response of wings in weak shear flow, in which the shear direction is normal to the wing surface.[8]

In this paper, an unsteady lifting surface theory based on the inviscid and incompressible spanwise shear flow model is presented. Although the small perturbation theory can be applied for this case, if the variation is small, it is impossible to use the velocity potential. However, a Poisson differential equation can be derived from the Euler equations; moreover, for the case of uniform shear, a Laplace equation for the acceleration can be obtained. By the method of collocations, generalized forces are calculated, and, finally, the unsteady response of wings is obtained for some kinds of shear.

Basic Equations

Consider a thin, rectangular wing in incompressible and inviscid flow illustrated in Fig. 1. Let u, v, and w denote perturbation velocities in the x, y, and z directions, respectively. The Euler equations of motion can be linearized as follows:

$$\frac{\partial u}{\partial t}+U(y)\frac{\partial u}{\partial x}+v\frac{\mathrm{d}U(y)}{\mathrm{d}y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0 \tag{1a}$$

$$\frac{\partial v}{\partial t}+U(y)\frac{\partial v}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial y}=0 \tag{1b}$$

$$\frac{\partial w}{\partial t}+U(y)\frac{\partial w}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial z}=0 \tag{1c}$$

where $U(y)$ is the freestream velocity distribution. Furthermore, the continuity equation is written by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{2}$$

The approaching stream is assumed to have a linear shear distribution given by

$$U(y)=U_0(1+y/e) \tag{3}$$

Although it is impossible to consider velocity potential in the shear flow, a partial-differential equation for the pressure or for one of the perturbation velocities can be obtained after differentiations and algebraic manipulations of Eqs. (1) and (2). This new equation is a Poisson type. In steady cases, as described previously, von Kármán and Tsien[3] and Honda[4] dealt with the equation for the pressure, and Lighthill[5] and Morita[6,7] considered the equation for the velocity v. In this investigation, we adopt the equation for v because the partial-differential equation becomes a Laplace equation due to the linear shear.

Consequently, the following differential equation can be derived:

$$\left[\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right]\nabla^2 v-\frac{\mathrm{d}^2U(y)}{\mathrm{d}y^2}\frac{\partial v}{\partial x}=0 \tag{4}$$

Since $\mathrm{d}^2U(y)/\mathrm{d}y^2=0$ by Eq. (3), Eq. (4) is written as a Laplace equation, namely,

$$\nabla^2 V(x,y,z,t)=0 \tag{5}$$

where V is the linearized perturbation acceleration expressed by

$$V=\left(\frac{\partial}{\partial t}+U(y)\frac{\partial}{\partial x}\right)v \tag{6}$$

Furthermore, assuming simple harmonic motion, all quantities can be written as

$$A(x,y,z,t)=\bar{A}(x,y,z)e^{i\omega t} \tag{7}$$

Then, Eqs. (5) and (6) are rewritten as

$$\nabla^2\bar{V}(x,y,z)=0 \tag{8}$$

where

$$\bar{V}=\left(i\omega+U(y)\frac{\partial}{\partial x}\right)\bar{v} \tag{9}$$

Integral Equation for the Lift Distribution

By Eqs. (1b) and (9), $\bar{p}$ is expressed by $\bar{V}$ as

$$\bar{p}(x,y,z)=-\rho\int_{-\infty}^{y}\bar{V}(x,y',z)\mathrm{d}y' \tag{10}$$

On the other hand, $\bar{w}$ can be solved from Eq. (1c), and, therefore, is expressed by $\bar{V}$ using Eq. (10) as follows:

$$\begin{aligned}\bar{w}(x,y,z)&=-\frac{e^{-i\frac{\omega x}{U(y)}}}{U(y)}\int_{-\infty}^{x}e^{i\frac{\omega\lambda}{U(y)}}\frac{1}{\rho}\frac{\partial\bar{p}(\lambda,y,z)}{\partial z}\mathrm{d}\lambda\\&=\frac{e^{-i\frac{\omega x}{U(y)}}}{U(y)}\int_{-\infty}^{x}e^{i\frac{\omega\lambda}{U(y)}}\int_{-\infty}^{y}\frac{\partial\bar{V}(\lambda,y',z)}{\partial z}\mathrm{d}y'\mathrm{d}\lambda\end{aligned} \tag{11}$$

Since the wing is assumed to be thin, the solution of the Laplace equation, Eq. (8), can be given by

$$\bar{V}(x,y,z)=-\frac{1}{4\pi}\iint_{S+W}\Delta\bar{V}(x',y')\frac{\partial}{\partial z}\left(\frac{1}{R}\right)\mathrm{d}x'\mathrm{d}y' \tag{12}$$

where $\Delta\bar{V}(x',y')=\bar{V}(x',y',+0)-\bar{V}(x',y',-0)$ and $R=\sqrt{(x-x')^2+(y-y')^2+z^2}$. On the other hand, $\Delta\bar{V}$ is expressed by the pressure difference as

$$\Delta\bar{V}(x,y,z)=\frac{1}{\rho}\frac{\partial\Delta\bar{p}(x,y)}{\partial y} \tag{13}$$

where $\Delta\bar{p}(x,y)=\bar{p}(x,y,-0)-\bar{p}(x,y,+0)$. Accordingly, the relation between $\bar{w}$ and $\Delta\bar{p}$, i.e., lift distribution l, is obtained through Eqs. (11-13) as follows:

$$\bar{w}(x,y,z)=-\frac{1}{8\pi}\frac{U_0^2}{U(y)}e^{-i\frac{\omega x}{U(y)}}\int_{-\infty}^{x}e^{i\frac{\omega\lambda}{U(y)}}\int_{-\infty}^{y}\times\iint_S\frac{\partial l(x',y')}{\partial y'}\frac{\partial^2}{\partial z^2}\left(\frac{1}{R_1}\right)\mathrm{d}x'\mathrm{d}y'\mathrm{d}y''\mathrm{d}\lambda \tag{14}$$

where

$$l(x',y')=\Delta\bar{p}(x',y')/\tfrac{1}{2}\rho U_0^2$$

$$R_1=\sqrt{(\lambda-x')^2+(y''-y')^2+z^2}$$

and the integration should be performed only on the wing surface as there is no pressure difference in the wake. Therefore, the integral equation for l can be obtained by changing the order of integration and taking the limit $z\rightarrow 0$ as follows:

$$\frac{\bar{w}_s(x,y)}{U_0}=-\frac{1}{8\pi}\iint_S\frac{\partial l(x',y')}{\partial y'}K(x,y;x',y')\mathrm{d}x'\mathrm{d}y' \tag{15}$$

where

$$\frac{\bar{w}_s(x,y)}{U_0}=\frac{\bar{w}(x,y,0)}{U_0}\frac{U(y)}{U_0} \tag{16}$$

$$K(x,y;,x',y')=\lim_{z\to 0}\frac{\partial^2}{\partial z^2}e^{-i\frac{\omega x}{U(y)}}\int_{-\infty}^{x}e^{i\frac{\omega\lambda}{U(y)}}\int_{-\infty}^{y}\frac{1}{R_1}\mathrm{d}y''\mathrm{d}\lambda \tag{17}$$

In the linear shear case, the kernel function, Eq. (17), is rewritten as

$$K(x,y;x',y')=\lim_{z\to 0}\frac{\partial^2}{\partial z^2}e^{-i\frac{\omega x_0}{U_0}\frac{1}{1+y/e}}\times\int_{-\infty}^{x_0}e^{i\frac{\omega\lambda}{U_0}\frac{1}{1+y/e}}\int_{-\infty}^{y}\frac{1}{R_2}\mathrm{d}y''\mathrm{d}\lambda \tag{18}$$

where $x_0=x-x'$, $R_2=\sqrt{\lambda^2+(y''-y')^2+z^2}$.

Method of Solution by Mode Functions

Comparing Eq. (15) with the integral equation of potential flow case, the sides of the equation differ to some extent from each other; that is, the left-hand side of Eq. (15) includes the velocity distribution $U(y)$ and the lift distribution is differentiated by η. Moreover, integration by η appears in the kernel function. In spite of these facts, the mode function method in lifting surface theory is applicable in order to solve the integral equation. Following Ref. 2, it is convenient to take

$$\bar{w}_s(x,y)=\exp(-i\bar{\nu}x)\bar{w}_s(x,y)$$
$$l(x',y')=\exp(-i\bar{\nu}x')\bar{l}(x',y') \tag{19}$$

with

$$\bar{l}(x',y')=\frac{8s}{\pi c(y')}\sum_{q=1}^{N}\{\Gamma_q(y')\Psi_q(\phi')\} \tag{20}$$

where

$$\bar{\nu}=\bar{\nu}(y)=\omega\bar{c}/U(y)\qquad x'=x_l(\eta')+½c(\eta')(1-\cos\phi')$$
$$\eta'=y'/s=-\cos\theta' \tag{21}$$

Then, the integral equation, Eq. (15), can be rewritten as

$$\frac{\bar{w}_s(x,\eta)}{U_0}=\frac{1}{2\pi}\sum_{q=1}^{N}\int_{-1}^{1}\frac{\mathrm{d}\Gamma_q(\eta')}{\mathrm{d}\eta'}H_q(x,\eta;x',\eta')\mathrm{d}\eta' \tag{22}$$

where

$$H_q(x,\eta;x',\eta')=-\frac{1}{\pi}\int_0^{\pi}e^{i\frac{\bar{\nu}(\eta)(x-x')}{\bar{c}}}\times K(x,\eta;x',\eta')\Psi_q(\phi')\sin\phi'\mathrm{d}\phi' \tag{23}$$

For a rectangular wing, it is possible to simplify Eq. (22) using integration by parts as follows:

$$\int_{-1}^{1}\frac{\mathrm{d}\Gamma_q(\eta'')}{\mathrm{d}\eta''}\int_{-\infty}^{\eta}\frac{1}{R_2}\mathrm{d}\eta'\mathrm{d}\eta''$$
$$=\Gamma_q(\eta'')\int_{-\infty}^{\eta}\frac{1}{R_2}\mathrm{d}\eta'\Big|_{-1}^{1}-\int_{-1}^{1}\Gamma_q(\eta'')\frac{\partial}{\partial\eta''}\times\int_{-\infty}^{\eta}\frac{\mathrm{d}\eta'\mathrm{d}\eta''}{\sqrt{\lambda^2+(\eta'-\eta'')^2+z^2}}$$
$$=\int_{-1}^{1}\Gamma_q(\eta'')\frac{1}{\sqrt{\lambda^2+(\eta-\eta'')^2+z^2}}\mathrm{d}\eta'' \tag{24}$$

Consequently, the integral equation for Γ_q, Eq. (22), is written as

$$\frac{\bar{w}_s(x,\eta)}{U_0}=\frac{1}{2\pi}\sum_{q=1}^{N}\int_{-1}^{1}\frac{\mathrm{d}\Gamma_q(\eta'')}{\mathrm{d}\eta''}H_q(x,\eta;x',\eta'')\mathrm{d}\eta'' \tag{25}$$

where

$$H_q(x,\eta;x',\eta'')=-\frac{1}{\pi}\int_0^{\pi}\exp\left\{\frac{i\bar{\nu}(\eta)(x-x')}{\bar{c}}\right\}\times K(x,\eta;x',\eta'')\Psi_q(\phi')\sin\phi'\mathrm{d}\phi' \tag{26}$$

In Eqs. (22) and (23), Ψ_q and Γ_q are pressure distribution functions along the chordwise and spanwise directions, respectively.

As described previously, the method of solution developed in Ref. 2 is applied (the details are omitted in this paper). If the mode function $\Gamma_q(\eta'')$ is interpolated by the following series,

$$\Gamma_q(\eta'')=\frac{2}{m+1}\sum_{r=1}^{m}\left\{\Gamma_{qr}\sum_{\mu=1}^{m}(\sin\mu\theta''\sin\mu\theta_\nu)\right\}$$
$$\theta_r=\frac{r\pi}{m+1} \tag{27}$$

the integral equation, Eq. (25), becomes a set of linear equations with Γ_{qr} as unknowns.

$$\frac{\bar{w}_s(x_{p\nu},\eta_\nu)}{U_0}=\sum_{q=1}^{N}\sum_{r=1}^{m}\Gamma_{qr}\Omega_q(p,\nu,r) \tag{28}$$

where

$$\Omega_q(p,\nu,r)=\sum_{\lambda=1}^{\Lambda+1}R_q(p,\nu,\lambda)\kappa_{r\lambda}+P_q(p,\nu)\rho_{r\nu}+P_q'(p,\nu)\sigma_{r\nu}+\left(\frac{s}{c_\nu}\right)^2E_q(p,\nu)\tau_{r\nu} \tag{29}$$

At this point, the wing surface is divided into small panels. These collocations are made as follows:

1) Chordwise direction:

$$x_{p\nu}=x_l(\eta_\nu)+½c(\eta_\nu)(1-\cos\phi_p),\quad \phi_p=\frac{2\pi p}{2N+1}$$
$$p=1,2,\ldots,N \tag{30}$$

2) Spanwise direction:

$$\eta_\nu=-\cos\theta_\nu,\quad \theta_\nu=\frac{\pi\nu}{m+1}\qquad \nu=1,2,\ldots,m \tag{31}$$

Several functions and coefficients which appear in Eq. (29) have the same forms in Ref. 2. On the other hand, the upwash

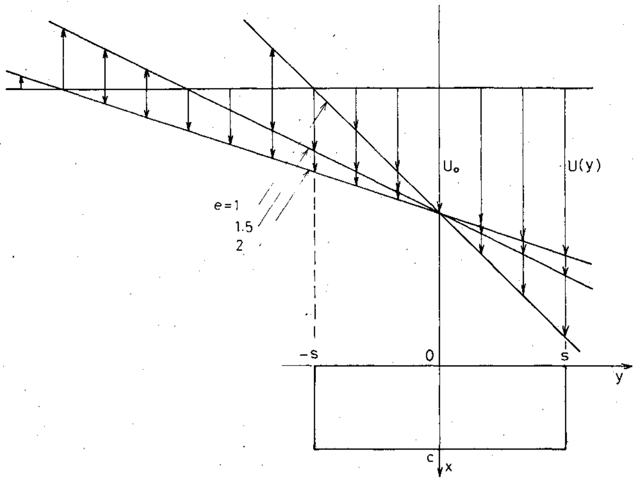

Fig. 2 Shear flow velocity distributions.

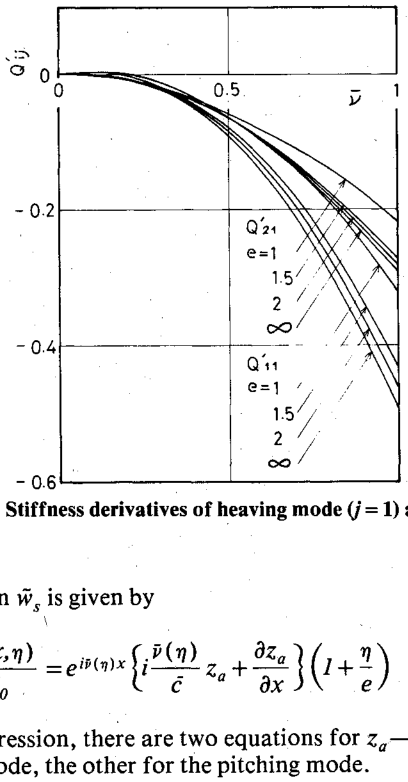

Fig. 3 Stiffness derivatives of heaving mode ($j=1$) at $AR=3$.

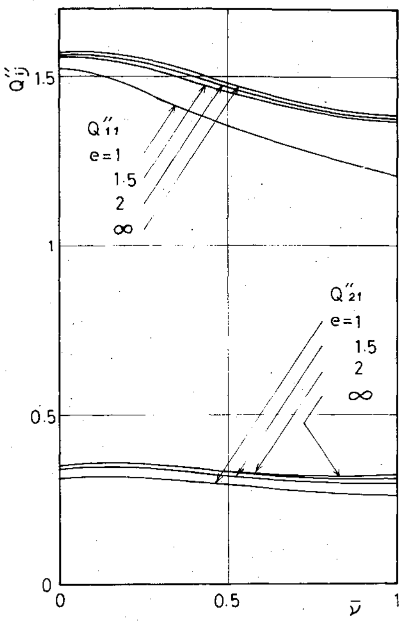

Fig. 4 Damping derivatives of heaving mode ($j=1$) at $AR=3$.

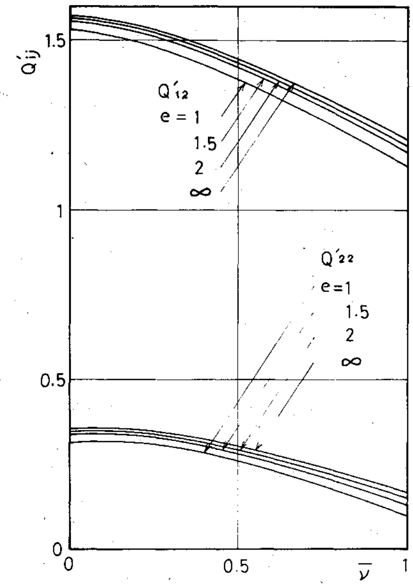

Fig. 5 Stiffness derivatives of pitching mode ($j=2$) at $AR=3$.

distribution $\tilde{w}_s$ is given by

$$\frac{\tilde{w}_s(x,\eta)}{U_0} = e^{i\bar{\nu}(\eta)x}\left\{i\frac{\bar{\nu}(\eta)}{\bar{c}}z_a + \frac{\partial z_a}{\partial x}\right\}\left(1+\frac{\eta}{e}\right) \tag{32}$$

In this expression, there are two equations for z_a—one for the heaving mode, the other for the pitching mode.

$$z_{a_1} = -\bar{c} \qquad z_{a_2} = -x \tag{33}$$

Numerical Calculations and Discussion

For the expression of unsteady response in shear flow, it is effective to calculate the generalized forces which are defined by

$$Q_{ij} = -\frac{1}{2dD}\iint_S z_i l_j \mathrm{d}x\mathrm{d}y \tag{34}$$

where i denotes the force mode and j the downwash mode, which corresponds with Eq. (33). Furthermore, this generalized force, which is a complex quantity, can be divided into real and imaginary parts.

$$Q_{ij} = Q'_{ij} + i\bar{\nu}Q''_{ij} \tag{35}$$

where Q'_{ij} and Q''_{ij} indicate the stiffness and damping derivatives, respectively. Numerical calculations are performed for rectangular wings of aspect ratios 3, 6, and 12 in this investigation. Programming techniques are mainly dependent on Ref. 2. The number of collocation points is $N=3$ in chordwise direction and $m=11$ in spanwise direction. Response functions to three amounts of shear are calculated and compared with each other and with the potential flow case. These velocity distributions are illustrated in Fig. 2. Among these distributions, $e=1$ is the limiting case, since the adverse flow region would appear on the wing surface if e becomes smaller than 1.

In Figs. 3-6, the generalized forces for the two oscillatory modes are shown for the wing of aspect ratio 3. The line of $e=\infty$ shows the potential flow case. From these figures, the effect of shear flow when $e=1.5$ and 2 is not as large as when $e=1$. However, the effect of shear flow in Q'_{11} and Q''_{12} appears even when $e=1.5$ and 2, and increases as $\bar{\nu}$ increases. Furthermore, the response functions of the wing to sinusoidal oscillations and a sinusoidal gust, which correspond to the Theodorsen function and the Sears function, respectively, are calculated. These functions can be obtained by substituting

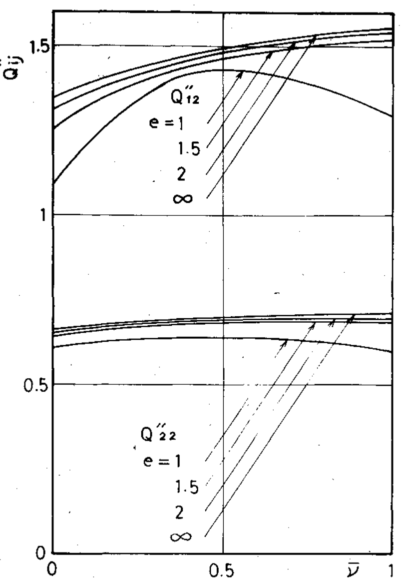

Fig. 6 Damping derivatives of pitching mode ($j=2$) at $AR=3$.

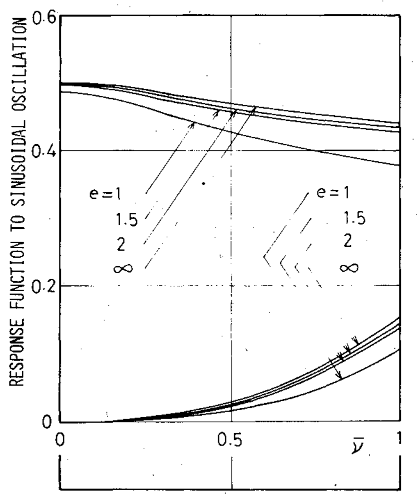

Fig. 7 Response functions to sinusoidal oscillation at $AR=3$.

$-e^{i\bar{\nu}x}\cdot\bar{\nu}(1+\eta/e)$ and $-(1+\eta/e)$ into the right-hand side of Eq. (32). The results are illustrated in Figs. 7 and 8 in which the aspect ratio is also 3. The effect of shear flow in the cases when $e=1.5$ and 2 is not as large in Fig. 7 as in Fig. 8; the effect of the shear flow is greater in the response to sinusoidal gust than in that to sinusoidal oscillation.

A similar tendency occurs in wings of different aspect ratios. In Fig. 9, the absolute values of the gust response function are illustrated against $1/e$. The values of the aspect ratio 3 and 6 decrease as e approaches 1, whereas that of 12 at first decreases, but then increases near $e=1$. This tendency agrees well with the results of the steady case.[7] However, as a whole, the effect of shear flow on the response is small in spite of large amounts of shear except for the case when $e=1$. The reason behind this fact is that since the loss and excess of flux over the left and right sides of the wing are equal with each other due to the linear shear, the difference between the pressure increase and decrease is not as large as expected. This can be easily seen if the lift distribution in heaving mode (Fig. 10) and the pitching moment in pitching mode (Fig. 11) along the spanwise direction are illustrated. In fact, the integrated values of the difference between the shear flow case and the potential flow case from $-s$ to s correspond to the effect of the shear flow, and it is clear from these figures that these values are small except for the case when $e=1$.

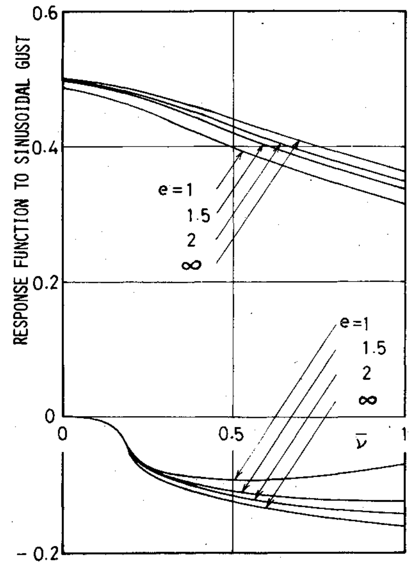

Fig. 8 Response functions to sinusoidal gust at $AR=3$.

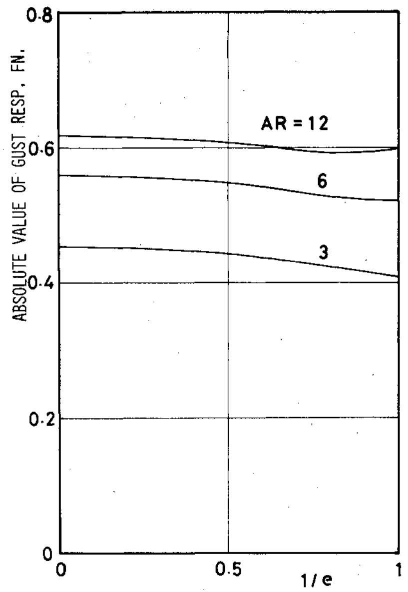

Fig. 9 Absolute values of gust response function at $\bar{\nu}=0.4$.

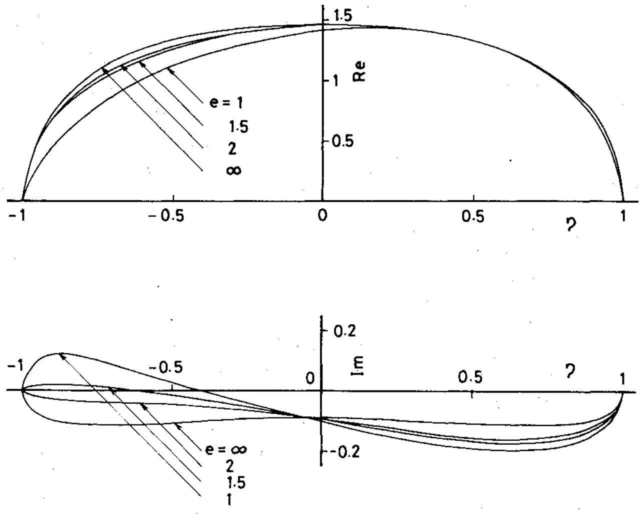

Fig. 10 Spanwise lift distributions of heaving mode ($j=1$) at $AR=3$, $\bar{\nu}=0.4$.

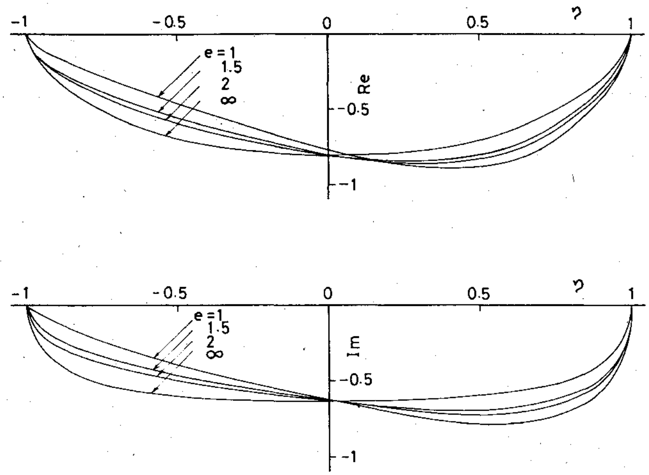

Fig. 11 Spanwise pitching moment distributions of pitching mode ($j=2$) at $Æ=3$, $\bar{\nu}=0.4$.

Conclusions

It is difficult to calculate the unsteady response of a wing in shear flow because it is impossible to use the velocity potential in this problem. It is especially difficult to solve this problem by the lifting surface theory. In this paper, it is noted that an integral equation which resembles that of potential flow in lifting surface theory can be obtained when the linear shear velocity distribution appears along the spanwise direction. Consequently, the lifting surface approach in potential flow is applicable for numerical calculations; however, the Mach number is restricted to zero.

As the response functions, generalized forces Q_{ij} defined by Eq. (34) are calculated. These functions express several kinds of forces and moments depending on the oscillation mode of the wing. Moreover, response functions to sinusoidal heaving oscillations and to a sinusoidal gust, which correspond to the Theodorsen function and the Sears function, respectively, are also calculated. The shear flow velocity distribution is expressed by Eq. (3) and amounts of shear are taken as $e=1$, 1.5, 2, and ∞ (the last value ∞ is the potential flow case). The results are compared with each other for several frequencies and show that the effect of shear is not large except when $e=1$, which is the limiting case shown in Fig. 2. However, the stiffness derivative Q'_{11}, the damping derivative Q''_{12}, and the gust response function are affected by the shear flow to some extent when $e=1.5$ and 2, and the effect increases as the frequency becomes larger.

Acknowledgment

The author wishes to express his gratitude to Prof. H. Försching for his support and encouragement during this research at DFVLR.

References

[1] Acum, W.E.A., "Theory of Lifting Surfaces Oscillating at General Frequencies in a Subsonic Stream," ARC Reports and Memoranda, R&M No. 3557, Feb. 1959.

[2] Lehrian, D. E. and Garner, H. C., "Theoretical Calculation of Generalized Force and Load Distribution on Wings Oscillating at General Frequencies in a Subsonic Stream," ARC Reports and Memoranda, R&M No. 3710, July 1971.

[3] von Kármán, T. and Tsien, H. S., "Lifting-Line Theory for a Wing in a Non-Uniform Flow," *Quarterly of Applied Mathematics*, Vol. III, No. 1, 1945, pp. 1-11.

[4] Honda, M., "Theory of a Thin Wing in a Shear Flow," *Proceedings of Royal Society*, Series A, Vol. 254, No. 1287, 1960, pp. 372-394.

[5] Lighthill, M. J., "The Fundamental Solution for Small Steady Three Dimensional Disturbances to a Two-Dimensional Parallel Shear Flow," *Journal of Fluid Mechanics*, Vol. 3, No. 2, 1957, pp. 113-144.

[6] Morita, K., "Theoretical Study of the Low-Aspect-Ratio Wing in Uniform Shear Flow," *Bulletin of the JSME*, Vol. 13, No. 57, 1970, pp. 362-372.

[7] Morita, K., "Lifting Line Theory of a Wing in Uniform Shear Flow," *Bulletin of the JSME*, Vol. 14, No. 72, 1971, pp. 550-563.

[8] Sasaki, A. and Kobayakawa, M., "Response of Oscillating Wings in Weak Shear Flow," *Transactions of the Japan Society for Aeronautics and Space Sciences*, Vol. 24, No. 65, 1981, pp. 139-151.